

Global Max/Min

Consider a surface $z = f(x,y)$ over a particular region R on the xy -plane.

An **absolute/global maximum** over R is the largest z -value over R .

An **absolute/global minimum** over R is the smallest z -value over R .

Key fact (Extreme value theorem)

The absolute max/min occur at either

1. A critical point, or
2. A boundary point.

Example: Let R be the triangular region in the xy -plane with corners at $(0,-1)$, $(0,1)$, and $(2,-1)$. Above this triangular region, find the absolute max and min of

$$f(x,y) = \frac{1}{4}x + \frac{1}{2}y^2 - xy + 1$$

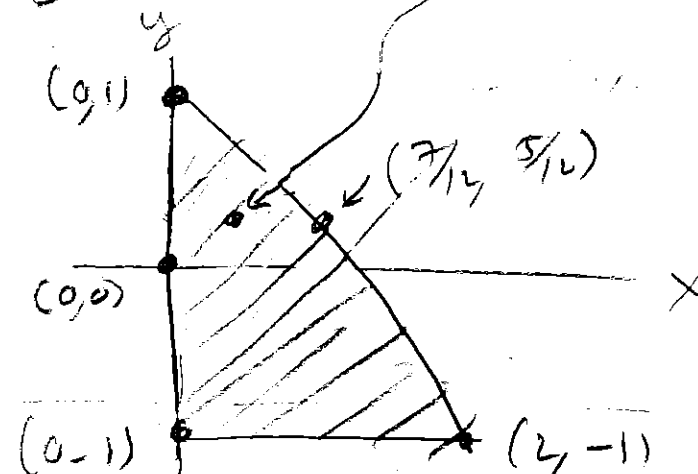
Entry Task

Do Step 1: Find the critical points

$$f_x = \frac{1}{4} - y \stackrel{?}{=} 0 \Rightarrow y = \frac{1}{4}$$

$$f_y = y - x \stackrel{?}{=} 0 \Rightarrow y = x$$

$$(x,y) = \left(\frac{1}{4}, \frac{1}{4}\right)$$



How to find the absolute max/min

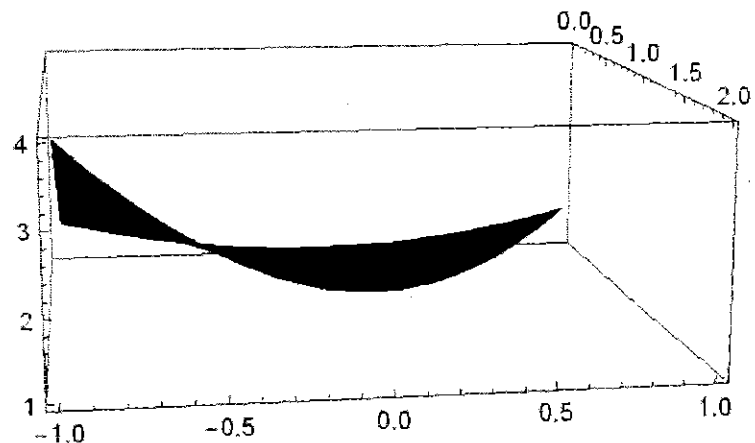
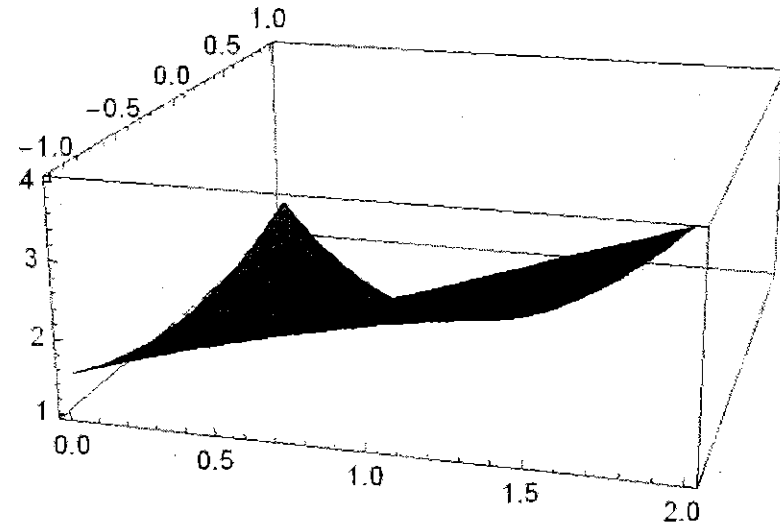
Step 1: Find critical points inside region.

Step 2: Find critical numbers and corners above each boundary.

- i) For each boundary, give an equation in terms of x and y . Find intersection with surface.
- ii) Find critical numbers and endpoints for this one variable function. Label "corners".

Step 3: Evaluate the function at all points you found in steps 1 and 2.

Biggest output = global max
Smallest output = global min



A $x=0$

$$z = f(0, y) = \frac{1}{2}y^2 + 1, \quad -1 \leq y \leq 1$$

$$z' = y \stackrel{?}{=} 0 \Rightarrow y = 0$$

NEED TO CONSIDER $y=0, y=-1, \text{ and } y=1$ ON THIS BOUNDARY.

ENDPOINTS (CORNERS)

A ONE VARIABLE ABS. MAX/MIN QUESTION!

B $y=-1$

$$z = f(x, -1) = \frac{1}{4}x + \frac{1}{2} + x + 1$$

$$z = -\frac{5}{4}x + \frac{3}{2}$$

$$0 \leq x \leq 2$$

$$z' = \frac{5}{4} \stackrel{?}{=} 0$$

NEED TO CONSIDER $x=0$ AND $x=2$ ON THIS BOUNDARY

AND $(2, -1) \Rightarrow$

$$m = \frac{-1 - (-1)}{2 - 0} = -1$$

C

LINE THRU $(0, 1)$

$y = -x + 1$

$$z = f(x, -x+1) = \frac{1}{4}x + \frac{1}{2}(-x+1)^2 - x(-x+1) + 1 = \frac{1}{4}x + \frac{1}{2}(x+1)^2 + x^2 - x + 1$$

$$z = -\frac{3}{4}x + \frac{1}{2}(x+1)^2 + x^2 + 1 \quad 0 \leq x \leq 2$$

$$z' = -\frac{3}{4} - (-x+1) + 2x \stackrel{?}{=} 0 \Rightarrow -\frac{7}{4} + 3x = 0 \Rightarrow x = \frac{7}{12}$$

$$x = \frac{7}{12} \Rightarrow y = -\frac{7}{12} + 1 = \frac{5}{12}$$

NEED TO CONSIDER $(\frac{7}{12}, \frac{5}{12})$ ON THIS SIDE (AND ENDPOINTS)

CONCLUSIONS

$$z = f(0, 1) = \frac{3}{2}$$

$$z = f(0, -1) = \frac{3}{2}$$

$$z = f(2, -1) = 4$$

$$z = f\left(\frac{1}{4}, \frac{1}{4}\right) = 1.03125$$

$$z = f\left(\frac{7}{12}, \frac{5}{12}\right) = 0.98958\bar{3}$$

$$z = f(0, 0) = 1$$

$$z = f(0, 0) = 1$$

$$\left(\frac{1}{2} + \frac{1}{2} + 2 + 1\right)$$

ABS. MAX

ABS. MIN

Example:

Find the absolute max/min of

$$f(x, y) = x^3 - 12x + y^2$$

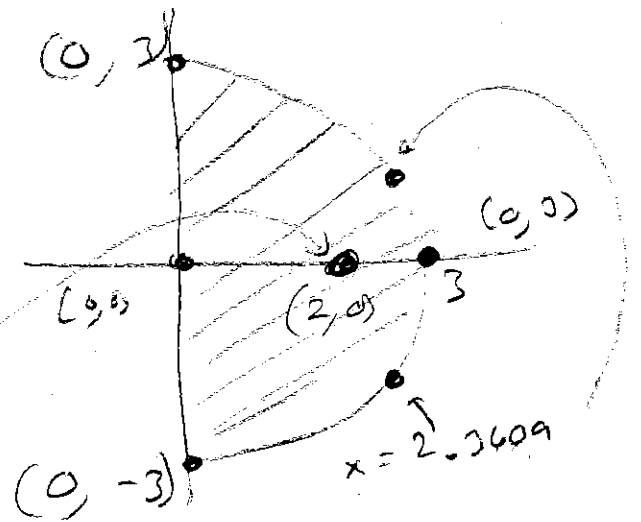
over the region

$$x \geq 0, x^2 + y^2 \leq 9.$$

$$f_x = 3x^2 - 12 \stackrel{!}{=} 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$f_y = 2y \stackrel{!}{=} 0 \Rightarrow y = 0$$

$(-2, 0)$ or $(2, 0)$
 OUTSIDE REGION



BOUNDARIES: [A] $x = 0 \Rightarrow z = f(0, y) = y^2 \quad -3 \leq y \leq 3$
 $z_y = 0 \Rightarrow y = 0$ (0, 0)

[B] $x^2 + y^2 = 9 \Rightarrow y^2 = 9 - x^2 \Rightarrow z = x^3 - 12x + 9 - x^2 \quad 0 \leq x \leq 3$
 $y = \pm \sqrt{9 - x^2}$
 $z' = 3x^2 - 12 - 2x \stackrel{!}{=} 0$
 $x = \frac{2 \pm \sqrt{4 - 4(7)(-12)}}{2(7)} = \frac{2 \pm \sqrt{320}}{14} \approx -1.6943$
 ≈ 2.3609

MAX & MIN RECALCULATE ONE OF THESE

$f(0, 3) = 9 \leftarrow \text{MAX}$
 $f(0, -3) = 9$
 $f(3, 0) = 27 - 36 = -9$

$f(2, 0) = 8 - 24 = -16 \leftarrow \text{MIN}$
 $f(2.3609, \sqrt{9 - (2.3609)^2}) \approx -11.745$
 $f(2.3609, -\sqrt{9 - (2.3609)^2}) \approx -11.745$

$f(0, 0) = 0$

Homework hints

In applied optimization problems,

- Identify what you are optimizing!
- Label Everything.
- Identify given facts (constraints)
- Use the constraints and labels to give a 2 variable function for the objective.

HW Examples:

- Find the points on the cone $z^2 = x^2 + y^2$ that are closest to $(4, 2, 0)$.

Objective: Minimize **distance** from (x, y, z) points on the cone to the point $(4, 2, 0)$ given that $z^2 = x^2 + y^2$.

$$\text{DIST TO } (4, 2, 0) = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

$$\text{CONSTRAINT: } z^2 = x^2 + y^2$$

$$\Rightarrow D(x, y) = \sqrt{(x-4)^2 + (y-2)^2 + x^2 + y^2}$$

Now

$$D_x = \frac{1}{2\sqrt{\quad}} (2(x-4) + 2x) \stackrel{?}{=} 0$$

$$D_y = \frac{1}{2\sqrt{\quad}} (2(y-2) + 2y) \stackrel{?}{=} 0$$

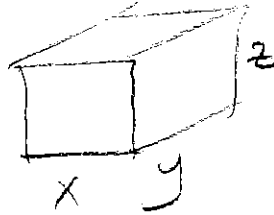
$$4x - 8 \stackrel{?}{=} 0 \Rightarrow x = 2$$

$$4y - 4 \stackrel{?}{=} 0 \Rightarrow y = 1$$

⋮

2. Find the dimensions of the box with volume 1000 cm^3 that has minimum surface area.

Objective: Minimize **surface area** given that volume is 1000.



$$\text{Surface Area} = 2xy + 2yz + 2xz$$

$$\text{CONSTRAINT: } xyz = 1000 \Rightarrow z = \frac{1000}{xy}$$

$$S(x,y) = 2xy + 2y \frac{1000}{xy} + 2x \frac{1000}{xy}$$

$$S(x,y) = 2xy + \frac{2000}{x} + \frac{2000}{y}$$

$$S_x = 2y - \frac{2000}{x^2} \stackrel{?}{=} 0$$

$$S_y = 2x - \frac{2000}{y^2} \stackrel{?}{=} 0$$

$$\Rightarrow y = \frac{1000}{x^2}$$

$$2x - \frac{2000}{\left(\frac{1000}{x^2}\right)^2} = 0$$

$$2x \left(\frac{(1000)^2}{x^4}\right) - 2000 = 0$$

$$\frac{(1000)^2}{x^3} = 1000$$

$$x^3 = 1000$$

$$x = (1000)^{1/3}$$

3. You want to build aquariums with slate for the base and glass for the sides (and no top).

Assume slate costs \$5 per in² and glass costs \$1 per in².

If the volume must be 1000 in³, then what dimensions will minimize cost?

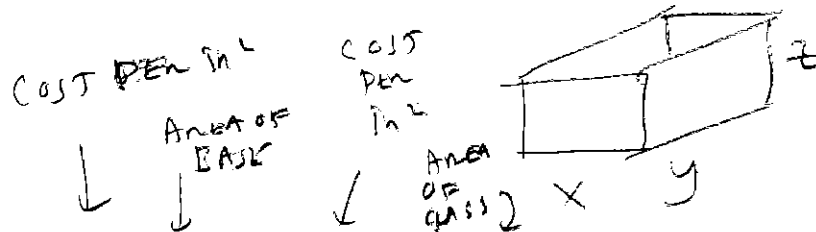
$$C(x,y) = 5xy + 2x \frac{1000}{xy} + 2y \frac{1000}{xy}$$

$$C(x,y) = 5xy + \frac{2000}{y} + \frac{2000}{x}$$

$$C_x = 5y - \frac{2000}{x^2} \stackrel{?}{=} 0$$

$$C_y = 5x - \frac{2000}{y^2} \stackrel{?}{=} 0$$

Objective: Minimize **cost** when volume needs to be 1000.



$$\text{COST} = 5xy + 1(2xz + 2yz)$$

$$xyz = 1000 \Rightarrow z = \frac{1000}{xy}$$